UW Math Circle, Winter 2017, Week 1

Number theory shmumber feary

This week we talked about classifying Pythagorean triples, the Pell equation, some other Diophantine equations, and how to go about solving these equations, or how to show that no solutions exist.

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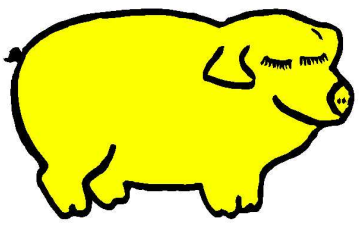
1. Show that there are at most finitely many Pythagorean triples with a given side length. For example, there are only finitely many integer triples $(5, b, c)$ with $b^2 + 5^2 = c^2$.
2. Find all Pythagorean triangles (that is, right triangles with integer sides) with the same area and perimeter.
3. (Hard) Show that a right triangle with integer sides can't have area equal to n^2 for any integer n.
4. We were able to solve the Pell equation $x^2 - 2y^2 = 1$ by finding a 'fundamental' solution $x = 3, y = 2$, and generating other solutions from that one. Can you find fundamental solutions to any other 'Pell equations' $x^2 - Ny^2 = 1$ for other values of N? (Hard:) Start with $N = 3, 4, 5, \ldots$: can you see any pattern? Is there always a unique 'fundamental' solution?
5. Which triangle numbers are square? Remember that the triangle numbers are the sums of the first n natural numbers for any n: the first few are $1, 3, 6, 10, 15, \ldots$. (Hint: this question can be re-written as a Diophantine equation. Does it remind you of an equation we've seen already?)
6. Show that if $x^n + y^n = z^n$ has no integer solutions when $n = 4$ or $n = p$ for any odd prime p, then Fermat's Last Theorem is true. (That is, that equation doesn't have any integer solutions for *any* n.)
7. Try playing the following game with a friend. Pick a number between 1 and 20, which you keep secret. Your friend can ask any question of the form 'does n divide your number?' for any number n, and you must answer honestly! What is your friend's best strategy to guess your number? How many questions does he need in the worst case to guess your number? (What if you pick between 1 and 1000?)
8. Here's a variant of the above game: again, one player picks a secret number between 1 and 1000, but now the questions the other player can ask are of the form 'what is the remainder upon division of your number by n?' for any $n < 50$. For example, if you pick 17 and your friend asks about division by 4, you would tell him that your number divided by 4 leaves a remainder of 1. Now what is the best strategy?



This is a picture of a pig. Can you tell how old the pig is?